Heterogeneous Multifrequency Direct Inversion in MR Elastography: A Preliminary Comparison of Finite-Difference and Finite-Element Based Approaches

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Overview

Outline

Collaborating Between Reconstruction Methods

- Many reconstruction methods have been published in PDE form (continuous case)
- *But*, reconstruction results depend on numerical details, which are rarely published
- A more unified and cross-validated approach to MRE reconstruction requires communicating about, and standardizing on, numerical details
- Only after that can the genuine differences between mathematically different reconstruction methods be well understood

Our Collaboration

Charité
Heterogeneous multifrequency reconstruction method, multifrequency *in vivo* acquisitions, pre-processing and denoising

KCL
Finite element reconstruction, finite element simulations, range of different reconstruction techniques (e.g. incorporation of pressure gradient, integration by parts)

Background

Isotropic linear elasticity equations, incompressible, time-harmonic form:

$$\rho\omega^2\mathbf{u} + \nabla\cdot\left(G\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) + \nabla p = \mathbf{0} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

In this iteration: Assume bulk wave is filtered out with a high-pass filter, leaving:

$$\rho\omega^2\mathbf{u} + \nabla\cdot\left(G\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) = \mathbf{0}, \tag{3}$$

and precalculate derivatives.

Now solve simultaneously across multifrequency acquisitions with a single matrix solve. This iteration: OLS.

Why Multifrequency 1/2: Eliminates Boundary Condition

1D case, first order ODE:

$$-\omega^2 u = (\mu u')' \tag{4}$$

General solution is

$$\mu(x)u'(x) = \mu(0)u'(0) - \omega^2 \int_0^x u(s)ds \tag{5}$$

But $\mu(0)$ is needed to have a solution. However we can eliminate $\mu(0)$ using two frequencies:

$$\begin{aligned} \frac{\mu(x)u_1'(x)}{u_1'(0)} &= \mu(0) - \omega_1^2 \int_0^x u_1(s)ds \\ \frac{\mu(x)u_2'(x)}{u_2'(0)} &= \mu(0) - \omega_2^2 \int_0^x u_2(s)ds \end{aligned} \tag{6}$$

Why Multifrequency 2/2: Better Matrix Conditioning

Again in 1D case, considering the linear system $A\mu = b$:

$$A = \begin{pmatrix} a_1 & -a_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & -a_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_3 & -a_4 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{N-1} & -a_N \end{pmatrix}$$

If any a_j are zero, A is no longer invertible. However when μ is $M \times N$ and A is $(\geq 2)M \times N$, projection of A onto space of μ may be invertible even with some missing a_j *without regularization*.

Why Direct: Better Estimate of First Order PDE

- *Inverse* elasticity problem is *first-order PDE*, as μ, μ' and λ, λ' are solved for.
- However, the *forward* elasticity problem is a *second order*, **elliptical** *PDE* as u'' is also present.
- First order PDEs may contain *jumps* or *discontinuities* in the field.
- Elliptical second-order PDEs are very smooth – smooth first derivatives!
- Consequently a *forward* estimation method produces heavily smoothed estimates of discontinuous solution spaces, while a *direct* method can estimate these discontinuities and avoid contamination of the solution by them (IF accompanied by the right numerics).

Outline

SFWI

The FD reconstruction utilizes second order difference stencils for all derivatives of (3) and constructs a matrix system to solve globally for the spatially variable G:

$$\begin{bmatrix} \nabla \cdot \epsilon_1 & \epsilon_1 \\ \nabla \cdot \epsilon_2 & \epsilon_2 \\ \dots & \\ \nabla \cdot \epsilon_n & \epsilon_n \end{bmatrix} \begin{bmatrix} I \\ \nabla^T \end{bmatrix} \mu = -\rho \begin{bmatrix} \omega_1 & & & \\ & \omega_2 & & \\ & & \dots & \\ & & & \omega_n \end{bmatrix}^2 \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \dots \\ \mathbf{u}_n \end{bmatrix} \tag{7}$$

FEM

The FEM solves the weak form of (3)

$$\int_\Omega \left(\rho\omega^2 \mathbf{u}\cdot\mathbf{w} - G\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) : \nabla\mathbf{w}\right) \mathrm{d}\Omega + \int_\Gamma G\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\cdot\mathbf{n}\cdot\mathbf{w}\,\mathrm{d}\Gamma = 0, \tag{8}$$

where the test function, **w**, is chosen to be zero on the boundary, Γ, so that the boundary integral is removed. Standard construction of FEM matrices, using quadratic hexahedral elements, leads to the discretized system

$$\mathbf{KG} = \rho\omega^2\mathbf{MU}. \tag{9}$$

Tikhonov regularization of the stiffness is employed by adding the Laplacian of G to the equations to be minimized. The final system is

$$\left(\mathbf{K}^T\mathbf{K} + \alpha_G \mathbf{D}_G^T\mathbf{D}_G\right)\mathbf{G} = \rho\omega^2\mathbf{K}^T\mathbf{MU} \tag{10}$$

after applying least squares and where $\mathbf{D}_G$ is a discrete Laplacian operator.

Materials

Subjects:

- Gel phantom
- Abdominal acquisition cohort
- Brain acquisition cohort

Image filtering scheme:

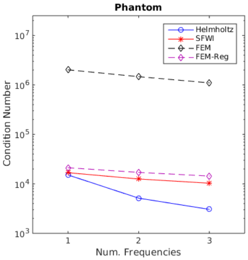
- Laplacian-Based Phase Unwrapping
- Denoising in complex dualtree wavelet basis with
- Overlapping Group Sparsity thresholding
- 3D 4th order Butterworth highpass filter (normalized cutoff $\omega = 0.03$)

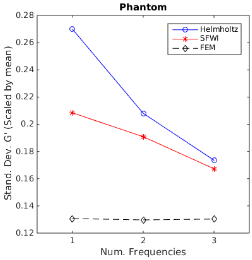
Outline

Multiple Frequencies Improves Matrix Conditioning

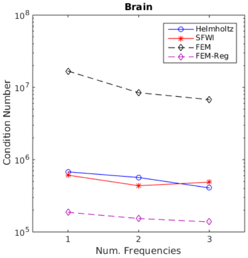
Condition Number | Standard Deviation

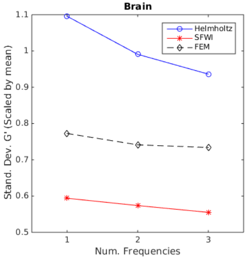
Phantom





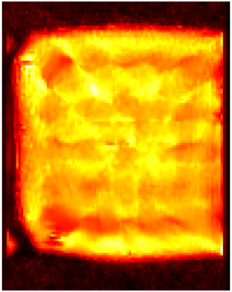
Brain

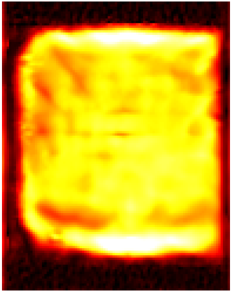


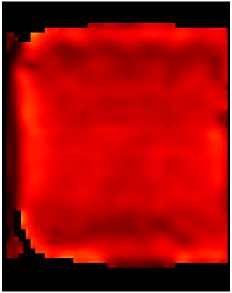


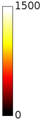
Reconstruction of gel phantom

Helmholtz	SFWI	FEM



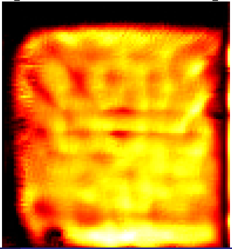




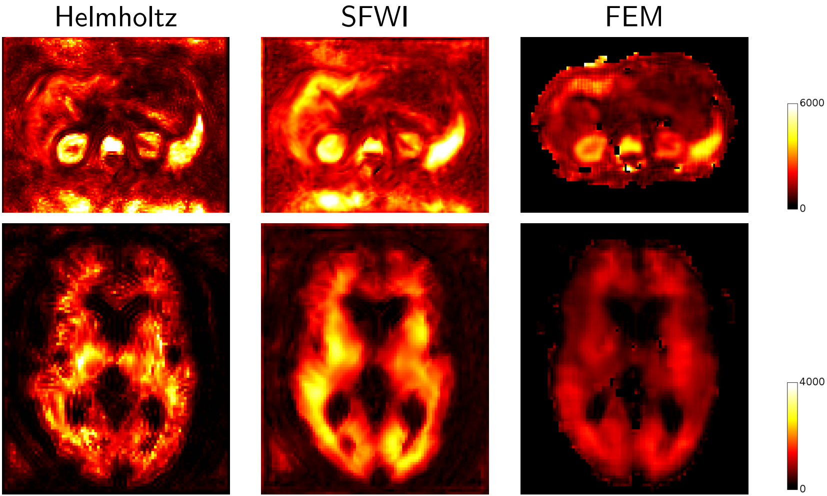


$\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$

$\begin{bmatrix} -0.5 & 0 & 0.5 \end{bmatrix}$



Other Preliminary Reconstructions



Abdomen: 30, 40, 50, 60 Hz. Brain: 30, 40, 50 Hz

Outline

Discussion

Some Early Observations

- Even minor changes to numerical implementation shift the result
- Least-squares solve is too smooth – TV-type penalties needed to sharpen it up

Future Work

- Compare with other N-L formulations: integration by parts, pressure term in solve (KCL)
- Sparsity-promoting penalties for solve: shearlet, smoothed-TV, TGV (EB)
- Finite volume implementations (EB)

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